

Sound Wave in Vortex with Sink

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Abstract

Using Komar's definition, we give expressions for the mass and angular momentum of a rotating acoustic black hole. We show that the mass and angular momentum so defined, obey the equilibrium version of the first law of Black Hole thermodynamics. We also show that when a phonon passes by a vortex with a sink, its trajectory is bent. The angle of bending of the sound wave to leading order is quadratic in A/cb and B/cb , where b is the impact parameter and A and B are the parameters in the velocity of the fluid flow. The time delay in the propagation of sound wave which to first order depends only on B/c^2 and is independent of A .

1 Introduction

In 1981, Unruh[1, 2] came up with the idea that the phenomenon of Hawking radiation could be observed in an inviscid fluid which has barotropic equation of state at sub-Planckian energy scales. The idea for this model arose from a very simple question as to how a sound wave propagates in an inhomogeneous fluid. He showed that if the fluid is barotropic and inviscid, and the flow of the fluid is irrotational, the equation of motion that the fluctuation of the velocity potential obeys is identical to that of a minimally coupled massless scalar field in curved space-time Lorentzian geometry. Since the

fluid has barotropic equation of state, pressure force does not generate vorticity. Thus an irrotational flow will remain irrotational. The viscosity of the fluid is ignored in order to maintain the Lorentzian character of the acoustic geometry. The fluid-flow will contain an analogue of a black hole if there is region where it flows with supersonic speed. The boundary of this acoustic black hole is a surface where the radial velocity of the fluid exceeds the local speed of sound. It has also been shown by Unruh that an acoustic black hole emits thermal radiation at a temperature given by the surface gravity at the sonic horizon. In his demonstration he has chosen a perfect sink flow ($\vec{v} = -A/r \hat{r}$) of the fluid which corresponds to non-rotating black hole at $r = A/c$ in the acoustic geometry. Later this idea was extended by Matt Visser to a more general situation where he considered a fluid motion which has both radial and azimuthal components. This fluid flow corresponds to rotating black hole at the same position in the acoustic geometry. This correspondence has been borne out by the fact that such an acoustic rotational black hole exhibits an analogue of superradiance as shown by Basak and Majumdar [3, 4]. As there is no analogue of the Einstein equation that the fluid under consideration obeys ¹, it can mimic only the kinematical aspects of gravity, not its dynamics.

For asymptotically flat solutions of Einstein's equation one can define the Komar charges corresponding to each isometry of the spacetime. In the case of spacetime around a Kerr black hole Komar charges corresponding to time translational and rotational Killing vectors are called the mass and angular momentum of the black hole and they are related by the equilibrium version of the first law of black hole thermodynamics. The basic criteria, for defining Komar charges and also for those charges to obey equilibrium version of the first law of black hole thermodynamics, have been discussed in section(3). Interestingly acoustic geometry can also satisfy all those criteria. Hence we can define conserved charges for acoustic black hole. Since the equilibrium version of the first law of black hole thermodynamics does not depend on Einstein's equation, Komar charges for the acoustic black hole will also satisfy this law.

One of the remarkable predictions of general relativity is that light trajectories are bent in passing by a massive object because the spacetime in the neighborhood of that massive object is curved. The bending of light in the neighborhood of a massive object can be described to sufficient accu-

¹The fluid motion is governed by the Euler equation and the continuity equation

racy using kinematical considerations alone since the effect of the light on the geometry is negligible. Moreover the source of curvature is not directly relevant to the discussion; the light bending effect is a direct consequence of the curved geometry which is locally Lorentzian. Since acoustic geometry is an example of Lorentzian geometry, it is expected that this phenomenon will also occur in this geometry if we consider a curved geometry arising from a suitable choice of velocity of the fluid flow.

Based on the idea that sound waves experience an effective gravitational field when passing through an inhomogeneous vorticity-free flow of a barotropic inviscid fluid, Fischer and Visser [6] showed that when a sound wave passes by an irrotational vortex, its trajectory is bent. This is analogous to the light bending effect in a physical gravitational field. In this paper we have demonstrated, using differential geometric techniques which we normally use in General Relativity, the bending of a sound wave for a spiral flow (vortex with sink) of the fluid. We have assumed that the background density of the fluid is constant, which implies that background pressure and the speed of sound are also constant and we have treated the sound wave as a ray. So our analysis is valid when $\frac{\lambda}{2\pi}$ is very less compared to the typical scale of variation of the acoustic geometry, but it should be greater than the interatomic distance because we have considered the fluid as a continuous medium.

The plan of the paper is as follows. We first briefly discuss in section (2), the geometry of an acoustic black hole. Using the geometric properties of acoustic space-time, we define the mass and angular momentum of an acoustic black hole in section (3). In section (4) we discuss the eikonal approximation which we will use to study the propagation of a sound wave in this geometry. In section (5) we present an analysis of null geodesics in acoustic geometry. This is followed by a demonstration of the bending of a phonon in acoustic geometry and the corresponding time delay in section (6) and (7) respectively. Experimental aspects have been discussed in section (8). Finally we conclude with a summary of the results obtained in this paper.

2 Acoustic Geometry

We consider the draining bathtub type of fluid motion with a line of sink at the center, which is basically a (2+1) dimensional flow with a sink at the origin. This leads to the velocity profile [7], [8] and [9] (in cylindrical polar

coordinates),

$$\vec{v} = -\frac{A}{r} \hat{r} + \frac{B}{r} \hat{\phi}, \quad (1)$$

where A and B are real and positive.

A two surface, at $r = \frac{A}{c}$ in this flow, where the fluid velocity is everywhere inward pointing and the radial component of the fluid velocity exceeds the local sound velocity everywhere, behaves as an outer trapped surface in this acoustic geometry. This surface can be identified with the future event horizon of the black hole. Since the fluid velocity (1) is always inward pointing, the linearized fluctuations originated in the region bounded by the sonic horizon cannot cross this boundary.

A linearized fluctuation in the dynamical quantities (density and pressure of the fluid, and velocity potential of the fluid motion) leads to the fluctuations of velocity potential and the equation of motion it satisfies is identical to that of a minimally coupled massless scalar field,

$$\frac{1}{\sqrt{-g}} \partial_\mu (\sqrt{-g} g^{\mu\nu} \partial_\nu) \Psi = 0 \quad (2)$$

where $g_{\mu\nu}$ is acoustic metric and it is given by,

$$ds^2 = \left(\frac{\rho_0}{c}\right)^2 \left[-\left(c^2 - \frac{A^2 + B^2}{r^2}\right) dt^2 + \frac{2A}{r} dr dt - 2B d\phi dt + dr^2 + r^2 d\phi^2 \right] \quad (3)$$

If we assume that the background density of the fluid is constant, it automatically implies that background pressure and the local speed of sound are also constant. Thus we can ignore the position independent pre-factor in the metric because it will not effect the equation of motion of fluctuations of the velocity velocity potential. As for the Kerr black hole in general relativity, the radius of the boundary of ergosphere of an acoustic black hole is given by vanishing of g_{00} , i.e, $r_{ergo} = \sqrt{A^2 + B^2}/c$.

3 Mass and Angular Momentum of Acoustic Black Hole

In this section I will discuss the definitions of certain conserved quantities in the case of a acoustic black hole which can be thought of as its mass

and angular momentum. This is identical to Komar's definition of mass and angular momentum using isometries of spacetime. In his work, almost 40 years ago, Komar[5] showed that for every isometry (ξ) of space time there exists a charge conserved(Q_ξ) on spatial hyper-surfaces(Σ) which is given by,

$$Q_\xi = \int_\Sigma {}^*d\xi \quad (4)$$

This conserved charge is arbitrary up to a constant factor, which can be fixed using a known solution to the Einstein equation. For a Kerr black hole the conserved charges corresponding to the time translational and rotational Killing vectors are the mass(M) and angular momentum(J) of the black hole and they are given by,

$$M_H = -\frac{1}{8\pi G} \int_H {}^*dk ; \quad J_H = \frac{1}{16\pi G} \int_H {}^*dm \quad (5)$$

where k and m are time translational and rotational Killing vectors respectively. The signs in equation(5) reflect the signature of the spacetime geometry.

Using this result Smarr[10] showed that if a stationary axisymmetric space-time contains a black hole then the mass and angular momentum of the black hole with respect to an observer at the horizon and one at asymptotic infinity are related as,

$$M = M_H - \frac{1}{4\pi G} \int_\Sigma {}^*R(k) \quad (6)$$

$$J = J_H + \frac{1}{8\pi G} \int_\Sigma {}^*R(m) \quad (7)$$

where ${}^*R(k)$ is Hodge dual of $R(k)(= R_{\mu\nu}k^\nu dx^\mu)$. M and J are mass and angular momentum of the black hole with respect to a stationary observer at infinity, and M_H and J_H are the mass and angular momentum of the black hole with respect to an observer at horizon. Since for vacuum solutions of the Einstein's equation, $R(k)$ and $R(m)$ vanish, the mass and angular momentum of a black hole with respect to observers at the horizon and at asymptotic infinity are the same. If the spacetime contains a Killing horizon such that the normal to the horizon is a linear combination of time translational Killing vector and rotational Killing vector as,

$$\ell = k + \Omega_H m \quad (8)$$

where Ω_H is the angular velocity of the horizon, then the mass at the horizon is given by [11],[12],

$$M_H = \frac{K_H A_H}{4 \pi G} + 2 \Omega_H J_H \quad (9)$$

If the black hole has electric charge then the above equation is modified to,

$$M = \frac{K_H A_H}{4 \pi G} + 2 \Omega_H J_H + V_H Q_H + \dots \quad (10)$$

where V_H is the electric potential at the horizon and Q_H is the charge of the black hole. If we restore all the occurrences of c , equation(9) is modified to the following form,

$$M_H = \frac{K_H A_H}{4 \pi G} + \frac{2 \Omega_H J_H}{c^2} \quad (11)$$

Even though in order to derive the equation(10) for charged black hole, one has to use Einstein's equation with energy-momentum tensor of electromagnetic fields, equation(11) is insensitive to the Einstein's equation. This equation entirely depends upon the geometric properties of the spacetime and does not depend upon how geometry of the spacetime is changing with time. The above equation will hold if a curved spacetime has the following properties,

- Geometry of the spacetime is pseudo-Riemannian with signature $(-, +, +, +)$
- Spacetime is asymptotically flat which implies that with increasing r , conserved current, $J^\mu = -R^\mu_\nu \xi^\nu$ decreases faster than the increase of area of the constant radius surface.
- Killing vectors of the spacetime geometry are time-translational and rotational.
- A Killing horizon is present with outward normal $l = k + \Omega_H m$

Acoustic geometry corresponding to the metric(3) is similar to the geometry of a curved spacetime containing a rotating black hole as far as kinematics are concerned. There are two isometries in this geometry and the corresponding Killing vectors are $k^\mu = \delta_t^\mu$ and $m^\mu = \delta_\phi^\mu$. The normal to the horizon is $l^\mu = k^\mu + \Omega_H m^\mu$. Since it is a linear combination of two Killing vectors

and $\Omega_H = \frac{B c^2}{A^2}$ is constant, this vector is also a Killing vector of the acoustic metric. Therefore the event horizon of the acoustic black hole is a Killing horizon. So we can define the surface gravity of that Killing horizon as,

$$l^\mu \nabla_\mu l_\nu = \frac{K_H}{c} l_\nu \quad (12)$$

where K_H is the surface gravity of the horizon which is equal to c^3/A . Hence this geometry is not only Lorentzian, but also satisfies all the four properties mentioned above. Therefore we can define the mass and angular momentum of the rotating acoustic black hole as the conserved charges corresponding to the time translational and rotational Killing vectors of the acoustic geometry. According to this, the conserved charges at the horizon corresponding to the time translational Killing vector is given by,

$$\begin{aligned} Q_k &= -\frac{c}{8\pi} \int_H {}^* dk \\ &= -\frac{c}{8\pi} \int_H \partial_\mu k_\nu \epsilon^{\mu\nu}{}_\alpha dx^\alpha \\ &= \frac{(A^2 + B^2) c^2}{2 A^2} \end{aligned} \quad (13)$$

and the conserved charge at the horizon corresponding to rotational Killing vector is given by,

$$\begin{aligned} Q_m &= \frac{c^3}{16\pi} \int_H {}^* dm \\ &= \frac{c^3}{16\pi} \int_H \partial_\mu m_\nu \epsilon^{\mu\nu}{}_\alpha dx^\alpha \\ &= \frac{B c^2}{4} \end{aligned} \quad (14)$$

These two quantities satisfy the following relation ,

$$Q_k = \frac{K_H L_H}{4\pi} + \frac{2\Omega_H Q_m}{c^2} \quad (15)$$

Here $L_H (= 2\pi \frac{A}{c})$ is the length of the horizon.

This equation is identical to the relation between mass and angular momentum of physical rotating black hole in $G = 1$ unit.

The most interesting feature of these results is that the angular momentum is proportional to the velocity parameter B . If one switches off the rotational motion of the fluid, B will become zero and hence the angular momentum of the black hole disappears. This is expected because in the absence of B , the horizon is no longer rotating. If the vortex motion is quantized, i.e, if B takes values proportional to an integer, then Q_m and Q_k take discrete values. From the expression of Q_m and Q_k it is clear that Q_m is proportional to integer and Q_k has a minimum value $c^2/2$. So we can say that this acoustic black hole is analogous to a physical black hole whose mass and angular momentum are quantized. Since there is no analog of Einstein's equation in acoustic analog models of gravity, the relation between the variation of Q_k and Q_m is not identical to that of physical process version of first law of black hole thermodynamics.

4 Eikonal Approximation

We now analyze the geometric properties of acoustic space-time in terms of the null geodesics. We will consider the propagation of a sound wave in this geometry as a ray. This is valid in the eikonal approximation [11],[13] which we describe briefly.

The eikonal approximation is described by the two conditions,

- Reduced wavelength($\frac{\lambda}{2\pi}$) of the sound wave is very small compared to the typical radius of curvature of the acoustic geometry.
- Reduced wavelength($\frac{\lambda}{2\pi}$) of the sound wave is very small compared to typical length over which the amplitude, polarization and wavelength of the wave change.

Under these conditions we can split the scalar field Ψ into a rapidly changing real phase and a slowly changing complex amplitude,

$$\begin{aligned}\Psi(x) &= Re \left\{ amplitude \times \exp \left(\frac{i}{\epsilon} \theta(x) \right) \right\} \\ &= Re \left\{ (a + \epsilon b + \epsilon^2 c + \dots) \times \exp \left(\frac{i}{\epsilon} \theta(x) \right) \right\}\end{aligned}\quad (16)$$

Here ϵ is a very small parameter. If R is the typical radius of curvature of the acoustic geometry and L is the typical length over which the amplitude and

the polarization of the wave remains unchanged, then the order of magnitude of ϵ is $\frac{\lambda}{2\pi}/\text{Min}(R, L)$. The first term (a) in the amplitude is wave length independent. It is the dominant term in the expression while the rest of the terms are wave length dependent and also very small compared to the first term. Mathematically, the scalar field (Ψ) corresponds to a massless particle in the acoustic geometry when its frequency tends to infinity, i.e., ($\lambda/2\pi \rightarrow 0$). Practically all the particles have finite de Broglie wave length. This causes the wavelength dependent corrections to the amplitude of the scalar field. These correction terms disappear in the $\lambda/2\pi \rightarrow 0$ limit.

Substituting this in equation(3) and then collecting terms of order $\frac{1}{\epsilon^2}$ and $\frac{1}{\epsilon}$ we get,

$$\nabla_\mu \theta \nabla^\mu \theta = 0 \quad (17)$$

and,

$$\nabla^\mu \theta \nabla_\mu a = -\frac{1}{2}(\nabla_\mu \nabla^\mu \theta) a \quad (18)$$

Since sound rays are defined as the curves normal to the constant phase $\theta(x)$ surface and the wave vector $k_\mu = \nabla_\mu \theta$ is normal to these surfaces, the trajectory of a sound ray is described by integral curves of wave vector.

According to this definition of wave vector(k),

$$k_\mu k^\mu = 0 \quad (19)$$

i.e, trajectories of sound ray are null curves and

$$k^\mu \nabla_\mu a = -\frac{1}{2}(\nabla_\mu k^\mu) a \quad (20)$$

Since the wave vector is the gradient of a scalar function, equation (19) implies that the integral curves of the wave vector are also null geodesics i.e,

$$k^\mu \nabla_\mu k^\nu = 0 \quad (21)$$

Equation(20) leads to two important results,

(1) If we define polarization of sound wave as,

$$f = \frac{a}{\sqrt{a\bar{a}}} \quad (22)$$

then,

$$f \bar{f} = 1 \quad (23)$$

and,

$$k^\mu \nabla_\mu f = 0, \quad (24)$$

So the polarization f is parallel transported along the integral curve of k^μ .

(2) We can define a conserved vector $j_\mu = a \bar{a} k_\mu$, satisfying

$$\nabla_\mu j^\mu = 0 \quad (25)$$

The corresponding conserved charge on spacelike hypersurfaces of acoustic geometry is,

$$Q = \int_\Sigma j^\mu d^3 \Sigma_\mu \quad (26)$$

Physically this conservation corresponds to conservation of sound rays (phonon number). Therefore in the eikonal approximation, we can treat the sound wave as a ray (phonon) which is a particle of zero rest mass with four momentum $p = \hbar k$ moving along null geodesics and parallel transporting the polarization f . Hence we will consider a phonon instead of sound wave in the analysis of null geodesics in acoustic geometry.

The curvature (Ricci Scalar) of the acoustic geometry is, $R = 2 r_{ergo}^2 / r^4$. Our analysis is valid if $\frac{\lambda}{2\pi} \ll r^2 / \sqrt{2} r_{ergo}$. Physically, when the wavelength of the phonon is less than the scale of variation of the metric, the effect of the curvature on the phonon's trajectory is significant.

5 Null Geodesics in Acoustic Geometry

As mentioned earlier there are two isometries (ξ_A) of the acoustic geometry. Corresponding to these we have the quantities conserved along the null geodesics given by,

$$K_A = -(\xi_A)^\mu g_{\mu\nu} \frac{dx^\nu}{d\lambda} \quad (27)$$

Here λ is the affine parameter of the geodesics and A takes values 1 and 2. Now using the components of the acoustic metric in the above equation we get the following equations,

$$K_1 = \left(c^2 - \frac{A^2 + B^2}{r^2} \right) \frac{dt}{d\lambda} - \frac{A}{r} \frac{dr}{d\lambda} + B \frac{d\phi}{d\lambda} \quad (28)$$

$$K_2 = B \frac{dt}{d\lambda} - r^2 \frac{d\phi}{d\lambda} \quad (29)$$

In order to find the expression for $\frac{dr}{d\lambda}$ we have made the following substitutions in equations,

$$dt = dT + \frac{A r}{r^2 c^2 - A^2} dr \quad (30)$$

$$d\phi = d\chi + \frac{A B}{r(r^2 c^2 - A^2)} dr \quad (31)$$

Due to these transformations, equations (28) and (29) are modified to following equations,

$$K_1 = \left(c^2 - \frac{A^2 + B^2}{r^2} \right) \frac{dT}{d\lambda} + B \frac{d\chi}{d\lambda} \quad (32)$$

$$K_2 = B \frac{dT}{d\lambda} - r^2 \frac{d\chi}{d\lambda} \quad (33)$$

Solving equations (32) and (33) we get,

$$\frac{dT}{d\lambda} = \frac{K_1 r^2 + K_2 B}{(r^2 c^2 - A^2)} \quad (34)$$

$$\frac{d\chi}{d\lambda} = \frac{K_1 B r^2 + K_2 (A^2 + B^2 - r^2 c^2)}{r^2 (r^2 c^2 - A^2)} \quad (35)$$

Since λ is the affine parameter of the null geodesics in acoustic geometry, the trajectory of a phonon can be determined by the following equation,

$$g_{\mu\nu} \frac{dx^\mu}{d\lambda} \frac{dx^\nu}{d\lambda} = 0 \quad (36)$$

Using the components of the acoustic metric in the above equation, we get,

$$\begin{aligned} \frac{r^2 c^2}{r^2 c^2 - A^2} \left(\frac{dr}{d\lambda} \right)^2 - \left(c^2 - \frac{A^2 + B^2}{r^2} \right) \left(\frac{dT}{d\lambda} \right)^2 - 2 B \frac{dT}{d\lambda} \frac{d\chi}{d\lambda} \\ + r^2 \left(\frac{dT}{d\lambda} \right)^2 + \left(\frac{dz}{d\lambda} \right)^2 = 0 \end{aligned} \quad (37)$$

Substituting $\frac{dT}{d\lambda}$ and $\frac{d\chi}{d\lambda}$ in equation(37) we get,

$$\left(\frac{dr}{d\lambda} \right)^2 = \frac{K_1^2}{c^2} + \frac{2 K_1 K_2 B}{r^2 c^2} - \frac{K_2^2}{r^2} \left(1 - \frac{A^2 + B^2}{r^2 c^2} \right) \quad (38)$$

Rearranging the above equation we get,

$$\frac{1}{2} \left(\frac{dr}{d\lambda} \right)^2 + V(r) = \frac{1}{2} \frac{K_1^2}{c^2} \quad (39)$$

where,

$$V(r) = \frac{K_2^2}{2 r^2} \left(1 - \frac{A^2 + B^2}{r^2 c^2} \right) - \frac{K_1 K_2 B}{r^2 c^2} \quad (40)$$

This equation describes the radial motion of a sound ray in acoustic geometry and it is identical to the one dimensional non-relativistic motion of a particle of unit mass and energy $\frac{K_1^2}{2 c^2}$ in a potential $V(r)$ [11].

The extrema of this potential, where $\frac{dV(r)}{dr} = 0$, are at,

$$r = \sqrt{\frac{2 K_2 (A^2 + B^2)}{K_2 c^2 - 2 B K_1}} \quad (41)$$

There is a difference in the nature of the effective potential $V(r)$ for $K_2 < 0$ as compared with for K_2 . If $K_2 < 0$ is negative then this potential has maxima at,

$$r_M = \sqrt{\frac{2 K_2 (A^2 + B^2)}{K_2 c^2 - 2 B K_1}} \quad (42)$$

for all negative values of K_2 because the 2nd derivative of this potential with respect to r is negative at $r = r_M$ for all negative values of K_2 . But when

$K_2 > 0$, $V(r)$ does not have any extrema (i.e, $\frac{dV(r)}{dr} = 0$, does not have any real root in r) if $K_2 c^2 \leq 2 B K_1$. Therefore when K_2 is positive, for the existence of a maximum in $V(r)$, we must have $K_2 c^2 > 2 B K_1$. If a sound wave moves towards the center of the vortex such that $V(r_M) > \frac{1}{2} \frac{K_1^2}{c^2}$, it will be reflected at the turning point where $\frac{dr}{d\lambda} = 0$. Considering the positive solutions of r of equation(39), we get position of the largest turning point of sound ray at,

$$r_T = \sqrt{\frac{K_2 (K_2 c^2 - 2 B K_1)}{2 K_1^2} \left(1 + \sqrt{1 - \frac{4 K_1^2 (A^2 + B^2)}{(K_2 c^2 - 2 B K_1)^2}} \right)} \quad (43)$$

If $b(= |K_2| c/K_1)$ is the magnitude of the impact parameter and $(A, B) \ll c b$ then expanding r_T in series (up to 2nd order) in the small quantities $\frac{A}{c b}$ and $\frac{B}{c b}$ we get,

$$r_T = b \left[1 - \frac{K_2}{|K_2|} \frac{B}{c b} - \frac{1}{2} \frac{A^2}{c^2 b^2} - \frac{B^2}{c^2 b^2} + \mathcal{O} \left\{ \left(\frac{A}{c b} \right)^3, \left(\frac{B}{c b} \right)^3 \right\} \right] \quad (44)$$

This shows that, when K_2 is negative (i.e angular momentum of the sound ray is positive), r_T is greater than impact parameter, b . Therefore in order to make the distance of closest approach shorter than the impact parameter we have to consider positive values of K_2 . Since $\frac{dr}{d\lambda}$ takes both positive and negative value, to find the bending angle we have to choose the sign of $\frac{dr}{d\lambda}$. The choice is made on the basis of fact that when $A = B = 0$, we should obtain the Newtonian result, i.e, duration of propagation sound wave through the fluid should be positive.

6 Bending of Phonon Trajectory

In this section we will calculate the angle of bending of a phonon ray, expanding around $A = 0$ and $B = 0$, as it passes by irrotational vortex. Before calculating the angle of bending one important thing has to be noticed. Since $\frac{d\phi}{dr}$ is not symmetric about the turning point even if $\frac{dr}{d\lambda}$ flips sign at that point, the change in the polar angle when the phonon travels towards the vortex is

not same as that when phonon travels away from the vortex. Therefore the total change of the polar angle throughout its path is given by,

$$\begin{aligned}\Delta\phi &= \int_{\infty}^{r_T} \frac{d\phi}{dr} dr + \int_{r_T}^{\infty} \frac{d\phi}{dr} dr \\ &= \Delta\phi_- + \Delta\phi_+\end{aligned}\tag{45}$$

Expanding $\Delta\phi_-$ around $A = B = 0$ we get,

$$\begin{aligned}\Delta\phi_- &= (\Delta\phi_-)_{A=B=0} + A \left(\frac{d}{dA}(\Delta\phi_-) \right)_{A=B=0} + B \left(\frac{d}{dB}(\Delta\phi_-) \right)_{A=B=0} \\ &+ \frac{A^2}{2} \left(\frac{d^2}{dA^2}(\Delta\phi_-) \right)_{A=B=0} + \frac{B^2}{2} \left(\frac{d^2}{dB^2}(\Delta\phi_-) \right)_{A=B=0} \\ &+ A B \left(\frac{d^2}{dA dB}(\Delta\phi_-) \right)_{A=B=0} + \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right\} \\ &= -\text{sgn}(K_2) \left[\frac{\pi}{2} + \frac{3}{8} \frac{\pi (A^2 + B^2)}{b^2 c^2} \right] - \frac{1}{2} \frac{A B}{b^2 c^2} \\ &+ \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right\}\end{aligned}\tag{46}$$

Doing the same for $\Delta\phi_+$ we get,

$$\begin{aligned}\Delta\phi_+ &= (\Delta\phi_+)_{A=B=0} + A \left(\frac{d}{dA}(\Delta\phi_+) \right)_{A=B=0} + B \left(\frac{d}{dB}(\Delta\phi_+) \right)_{A=B=0} \\ &+ \frac{A^2}{2} \left(\frac{d^2}{dA^2}(\Delta\phi_+) \right)_{A=B=0} + \frac{B^2}{2} \left(\frac{d^2}{dB^2}(\Delta\phi_+) \right)_{A=B=0} \\ &+ A B \left(\frac{d^2}{dA dB}(\Delta\phi_+) \right)_{A=B=0} + \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right\}\end{aligned}$$

$$\begin{aligned}
&= -\text{sgn}(K_2) \left[\frac{\pi}{2} + \frac{3}{8} \frac{\pi (A^2 + B^2)}{b^2 c^2} \right] + \frac{1}{2} \frac{A B}{b^2 c^2} \\
&+ \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right\}
\end{aligned} \tag{47}$$

Hence the total change of polar angle of a phonon trajectory is,

$$\Delta\phi = -\text{sgn}(K_2) \left[\pi + \left\{ \frac{3}{4} \pi (A^2 + B^2) \right\} \frac{1}{b^2 c^2} \right] + \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right\} \tag{48}$$

The first term corresponds to the case when fluid velocity $\vec{v} = 0$. This matches with the Newtonian result. We have considered terms to leading order in A and B , which in this case is the 2nd order, but the terms corresponding to AB for $\Delta\phi_-$ and $\Delta\phi_+$ cancel each other and hence do not appear in the expression. Physically when the phonon approaches the turning point its motion is aided by the radial fluid motion, while it is hindered after crossing the turning point. The actual angle of bending is given by,

$$\begin{aligned}
\delta\phi &= \Delta\phi - \Delta\phi \Big|_{A=B=0} \\
&= -\text{sgn}(K_2) \left[\frac{3}{4} \pi (A^2 + B^2) \right] \frac{1}{b^2 c^2} + \mathcal{O} \left[\left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right] \\
&= -\text{sgn}(K_2) \frac{3}{4} \pi \left(\frac{r_{ergo}}{b} \right)^2 + \mathcal{O} \left[\left(\frac{A}{cb} \right)^3, \left(\frac{B}{cb} \right)^3 \right]
\end{aligned} \tag{49}$$

This is consistent with Fischer and Visser's[6] result for an irrotational vortex without a sink. Here the integrals are slightly tricky because the radius of the turning point depends both on A and B . So when we differentiate $\Delta\phi$ with respect to A and B , we take this into account. We have done the integration from r_T to ∞ . This is a good approximation because the radius of the boundary of the vortex is very large compared to the turning point.

7 Time delay and advance

Unlike in the case of the bending of a phonon, in order to calculate the time delay and time advance[14, 15, 16, 17, 18, 19] in phonon propagation, we will

compare null geodesics of same turning point for the fluid with motion and the fluid at rest. Using equation (43) we express K_2 in terms of r_T ,

$$K_2 = \frac{K_1 r_T^2 \left(B \pm \sqrt{c^2 r_T^2 - A^2} \right)}{(c^2 r_T^2 - A^2 - B^2)} \quad (50)$$

Substituting K_2 in $\frac{dr}{d\lambda}$ and $\frac{dt}{d\lambda}$ we get, time interval in which a phonon passes through the vortex,

$$\begin{aligned} \Delta t &= \int_{r_V}^{r_T} \frac{dt}{dr} dr + \int_{r_T}^{r_V} \frac{dt}{dr} dr \\ &= \Delta t_- + \Delta t_+ \end{aligned} \quad (51)$$

where r_V is the radius of the vortex.

Expanding Δt_- around $A = B = 0$ for fixed r_T we get,

$$\begin{aligned} \Delta t_- &= (\Delta t_-)_{A=B=0} + A \left(\frac{d}{dA} (\Delta t_-) \right)_{A=B=0} + B \left(\frac{d}{dB} (\Delta t_-) \right)_{A=B=0} \\ &+ \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^2, \left(\frac{B}{cb} \right)^2 \right\} \\ &= \frac{1}{c} \sqrt{r_V^2 - r_T^2} + \text{sgn}(K_2) \frac{B}{c^2} \tan^{-1} \left(\sqrt{\left(\frac{r_V}{r_T} \right)^2 - 1} \right) \\ &- \frac{A}{c^2} \ln \left(\frac{r_V}{r_T} \right) + \mathcal{O} \left\{ \left(\frac{A}{c^2} \right)^2, \left(\frac{B}{c^2} \right)^2 \right\} \end{aligned} \quad (52)$$

Doing the same for Δt_+ we get,

$$\begin{aligned} \Delta t_+ &= (\Delta t_+)_{A=B=0} + A \left(\frac{d}{dA} (\Delta t_+) \right)_{A=B=0} + B \left(\frac{d}{dB} (\Delta t_+) \right)_{A=B=0} \\ &+ \mathcal{O} \left\{ \left(\frac{A}{cb} \right)^2, \left(\frac{B}{cb} \right)^2 \right\} \\ &= \frac{1}{c} \sqrt{r_V^2 - r_T^2} + \text{sgn}(K_2) \frac{B}{c^2} \tan^{-1} \left(\sqrt{\left(\frac{r_V}{r_T} \right)^2 - 1} \right) \\ &+ \frac{A}{c^2} \ln \left(\frac{r_V}{r_T} \right) + \mathcal{O} \left\{ \left(\frac{A}{c^2} \right)^2, \left(\frac{B}{c^2} \right)^2 \right\} \end{aligned} \quad (53)$$

In contrast to the angle of bending, the time delay to leading order is linear in A and B . If we assume that $(A/c^2\tau, B/c^2\tau) \ll 1$, where $\tau = \frac{1}{c}\sqrt{r_V^2 - r_T^2}$, we can neglect the higher order terms. Subtracting the zeroth order term from Δt we get the time delay up to first order in A and B to be,

$$\begin{aligned}\delta t &= \Delta t - \Delta t \Big|_{A=B=0} \\ &= \text{sgn}(K_2) \frac{2B}{c^2} \tan^{-1} \left(\sqrt{\left(\frac{r_V}{r_T}\right)^2 - 1} \right) + \mathcal{O} \left\{ \left(\frac{A}{c^2}\right)^2, \left(\frac{B}{c^2}\right)^2 \right\} \quad (54)\end{aligned}$$

The only difference is in the limits of integration. This is because of the fact that the zeroth order term blows up if we put infinity in place of the radius of the vortex. In fact, the radius of the vortex, in actual experiment is large but finite. So a phonon will pass through the vortex in a finite time interval. The effect of the radial flow of the fluid on the propagation of the phonon is different during the two stages of motion – before and after having crossed the turning point. When the phonon approaches the turning point its motion is aided by the radial fluid motion, while it is hindered after crossing the turning point. The effect of the rotational velocity of the fluid is the same throughout the motion. Depending on the sign of K_2 and the angular momentum of the fluid, the sign of the time “delay” is decided by the combined effect of these. The terms that are first order in A/c^2 of Δt_+ and Δt_- cancel each other. This is because of the fact that when the radial coordinate is decreasing the radial flow of the fluid favors its motion and in the opposite case it opposes its motion.

8 Experimental Outlook

Unlike Hawking radiation and Superradiance for gravitational black-holes, the phenomena of bending of light and the corresponding time delay, as it passes by a massive object, have been experimentally observed. Hence experimental confirmation of the bending of the trajectory of a phonon will give further credence to our view that the phenomena of bending of the trajectory of a massless particle (and the corresponding time delay) in curved geometries, are features that are generic to Lorentzian geometry and independent of the dynamics of the system under consideration. In fact it would liberate

these phenomena from the almost stereotypical association with gravitating systems alone. For a given de Broglie wavelength of a phonon, there is a restriction on its impact parameter. If the impact parameter of a phonon is smaller than its de Broglie wavelength then the phonon behaves quantum mechanically and hence both the concepts of trajectory and impact parameter are meaningless. On the other hand, classical analysis of phonon propagation can be done, when the de Broglie wavelength is very very small compared to the impact parameter. So, if we consider the semiclassical scattering of phonon in acoustic geometry, the minimum value of the impact parameter should be equal to the de Broglie wavelength of a phonon. Therefore the maximum value of the angle of bending (up to 2nd order) of the phonon trajectory and the corresponding time delay (up to 1st order) are given by,

$$\delta\phi|_{max} = \frac{3\pi}{4} \left(\frac{r_{ergo}}{\lambda_{ph}} \right)^2 \quad (55)$$

$$\delta t|_{max} = \frac{2B}{c^2} \tan^{-1} \left(\sqrt{\left(\frac{r_V}{\lambda_{ph}} \right)^2 - 1} \right) \quad (56)$$

where λ_{ph} is the wave-length of phonon.

Since vortex motion with a sink has not yet been realized in the laboratory, it is not possible to give a correct estimate of the angle of bending and the corresponding time delay, but it is certainly possible if we consider perfect vortex flow ($A=0$). For a perfect vortex flow equations (55) and (56) are modified to,

$$\delta\phi|_{max} = \frac{3\pi}{4} \left(\frac{r_{ergo}}{\lambda_{ph}} \right)^2 \quad (57)$$

$$\delta t|_{max} = \frac{2B}{c^2} \tan^{-1} \left(\sqrt{\left(\frac{r_V}{\lambda_{ph}} \right)^2 - 1} \right) \quad (58)$$

where $r_{ergo} = \frac{B}{c}$.

Instead of a single phonon if we consider two phonons propagating in the same direction but on opposite sides of the vortex, they will be seen to converge to a point. Thus a vortex in an inviscid, barotropic fluid acts like a

convergent lens and the corresponding focal length(up to 2nd order) is given by,

$$f_{min} = \frac{2}{3\pi} \frac{\lambda_{ph}^3}{r_{ergo}^2} \quad (59)$$

The circulation for a perfect vortex flow ($\vec{v} = B/r \hat{\phi}$) is defined as,

$$\Gamma = \oint \vec{v} \cdot d\vec{x} = 2\pi B \quad (60)$$

If we consider a singly quantized vortex($B = \hbar/m$), then the circulation of the vortex is, $\Gamma = \frac{\hbar}{m}$.

As discussed in [9] there are two cases where we see perfect vortex motion,

1. Superfluid Helium, 4He ,
2. Bose-Einstein condensation in atomic gases.
 - (a) Bose-Einstein condensation of ^{23}Na ,
 - (b) Bose-Einstein condensation of ^{87}Rb ,

(1) **Superfluid Helium, 4He** : In case of superfluid helium[20], [21], [22],[23], [24] the energy spectrum of the quasiparticle, as shown by Feynman, is,

$$\epsilon(k) = \frac{\hbar^2 k^2}{2 m S(k)} \quad (61)$$

where $S(k)$ is the structure factor of the liquid which can be determined from neutron scattering. For small k , $\epsilon(k)$ is linear and for $k = 1.92 \text{ \AA}^{-1}$, $S(k)$ has a maximum. This corresponds to a minimum in $\epsilon(k)$. The excitations around that minimum are called rotons and those corresponding to the linear part of the spectrum are called phonons. In case of superfluid helium the roton minimum occurs at wave length $\lambda = 3.27 \text{ \AA}$ and the spectrum is phononic in the wavelength range roughly three times of it. At temperature very close to $0^\circ K$, the speed of sound in helium is 240 m/s. So for a singly quantized vortex in liquid Helium,

- Wavelength of phonon (λ_{ph}) = 9.81 \AA .
- Radius of the ergo sphere, (r_{ergo}) = 0.655 \AA .
- Maximum angle of bending of phonon, $\delta\phi|_{max} = 0.6^\circ$

- Minimum focal length, $f|_{min} = 467 \text{ \AA}$
- Maximum time delay is, $\delta t|_{max} = 0.857 \text{ ps}$.

(2) **Bose – Einstien Condensate** : In case of a dilute Bose-Einstein condensate [26, 27, 28, 29, 30, 31, 32] the energy spectrum of the quasiparticle excitation is of the Bogoliubov type and is given by,

$$\epsilon_k = \sqrt{(\epsilon_k^0)^2 + 2 n U_0 \epsilon_k^0} \quad (62)$$

where $\epsilon_k^0 = \frac{\hbar^2 k^2}{2m}$, $U_0 = \frac{4\pi\hbar^2 a}{m}$ is the strength of the effective interaction between the atoms, n is the density of atoms and m is the mass of the atoms. This spectrum is not similar to that of superfluid helium. In superfluid helium the quasiparticle excitations exhibit roton minimum because of strong short range correlation between the helium atoms, but due to the diluteness of the atomic vapor quasiparticle excitation spectrum does not exhibit roton minimum. The Bogoliubov spectrum is linear in the low wavenumber limit and quadratic in the large wavenumber limit. The transition from the linear to quadratic spectrum occurs when $\epsilon_k^0 \sim nU_0$. The reduced wavelength of the phonon corresponding to this transition point is called coherence length(ξ). For wavelengths less than the coherent length of the quasiparticle the spectrum is phononic. Since the velocity of sound is $c = \sqrt{nU_0/m}$, the coherent length(ξ) = $\hbar/\sqrt{2}mc$. For a singly quantized vortex $r_{ergo} = \hbar/mc$, so $2\pi\xi = \lambda = \sqrt{2}\pi r_{ergo}$.

(a) **BEC of Sodium, ^{23}Na** : In experiments conducted by the MIT group, a BEC of sodium atoms was produced with the following parameters:

- Number density of Sodium atoms, $(n) = 4 \times 10^{20} \text{ m}^{-3}$.
- Speed of sound, $(c) = 10.4 \text{ mm/s}$.
- Coherence length (ξ) = $0.187 \text{ }\mu\text{m}$.

Using the above data the following are calculated to be:

- Wavelength of phonon (λ_{ph}) = $1.172 \text{ }\mu\text{m}$.
- Radius of the ergo sphere, $(r_{ergo}) = 0.263 \text{ }\mu\text{m}$.

- Maximum angle of bending of phonon, $\delta\phi\big|_{max} = 6.84^\circ$
- Minimum focal length, $f\big|_{min} = 4.91 \mu m$
- Maximum time delay is, $\delta t\big|_{max} = 80 \mu s$.

(b) **BEC of Rubidium, ^{87}Rb** : We also have from the JILA experiment using Rubidium:

- Number density of Rubidium atoms, $(n) = 2.6 \times 10^{20} m^{-3}$.
- Speed of sound, $(c) = 3.0 mm/s$.
- Coherence length $(\xi) = 0.165 \mu m$.

Using this data-set the calculated values are found to be:

- Wavelength of phonon $(\lambda_{ph}) = 1.037 \mu m$.
- Radius of the ergo sphere, $(r_{ergo}) = 0.233 \mu m$.
- Maximum angle of bending of phonon, $\delta\phi\big|_{max} = 6.84^\circ$
- Minimum focal length, $f\big|_{min} = 4.37 \mu m$
- Maximum time delay is, $\delta t\big|_{max} = 0.237 ms$.

Looking at the order of magnitude of the observable, it is evident that BEC in atomic gases are more suitable than liquid helium for experimental verification of the bending of a phonon trajectory and the corresponding time delay. Even though the order of magnitude of the observables for the BEC is small, given the recent advances in technology, they are detectable.

9 Discussion

In acoustic analogue models of gravity, propagation of a sound wave in an inhomogeneous flow of barotropic inviscid fluid is equivalent to that of a minimally coupled massless scalar field in curved acoustic geometry which is Lorentzian in signature. In the eikonal limit we can treat a sound wave as a ray and also consider only the propagation of a single phonon instead of a sound ray in the acoustic geometry. In this limit the trajectory of a phonon is a null geodesic in the acoustic geometry. When such a phonon travels through an inhomogeneous flow of a barotropic inviscid fluid, its trajectory is bent. From the perspective of acoustic geometry this is the bending of the trajectory of a massless particle due to the curvature of the acoustic geometry. Due to this bending of the trajectory, a phonon has to travel along a path which is longer than the path that it would have taken had the fluid been at rest. In the short wavelength limit, the leading terms in the expansion of the bending angle in powers of the velocity parameters, are quadratic. This result is consistent with Fischer and Visser's[6] result for an irrotational vortex without a sink. In the same limit the expansion for the time delay in powers of the velocity parameters (A and B) to first order, is independent of A . This phenomenon in acoustic geometry is similar to the light bending effect and the corresponding time delay in a physical gravitational system when light passes by a massive object.

Instead of a single phonon if we consider two phonons propagating in the same direction but on opposite sides of the vortex, they will be seen to converge to a point. Thus a vortex in an inviscid, barotropic fluid acts like a convergent lens. Both the angle of bending and the time delay in the case of a BEC are significantly larger than those for liquid helium. In this respect BEC's in atomic gases are more suitable than liquid helium for experimental verification of these effects. Since this acoustic analog model of gravity is based on the assumption that when a sound wave passes through a flowing fluid, the linearized perturbations of dynamical quantities are stable, demonstrating the stability of the vortex under such perturbations is more important an issue than concerns about the order of the magnitude of the observable. This is one of the directions in which this work can be extended in future.

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